ON THE LOCAL SMOOTHING FOR A CLASS OF CONFORMALLY INVARIANT SCHRÖDINGER EQUATIONS

LUIS VEGA AND NICOLA VISCIGLIA

Abstract

We present some a - priori bounds from above and from below for solutions to a class of conformally invariant Schrödinger equations. As a by - product we deduce some new uniqueness results.

1. Introduction

The main aim of this paper is to extend a previous result proved in [12] about the local smoothing for the free Schrödinger equation, to a more general class of Schrödinger equations (linear and semilinear) that are invariant under the conformal transformation.

More precisely we shall consider the following Cauchy problems:

$$\mathbf{i}\partial_t u - \Delta u + |x|^{-2} W\left(\frac{x}{|x|}\right) u = 0, \tag{1.1}$$
$$u(0) = f, (t,x) \in \mathbf{R} \times \mathbf{R}^n, n \geq 3,$$

where $W : \mathbf{S}^{n-1} \to \mathbf{R}$ is a non - negative, bounded and measurable function (see also remark 1.4) and

$$\mathbf{i}\partial_t u - \Delta u \pm u|u|^{\frac{4}{n}} = 0, \tag{1.2}$$
$$u(0) = f, (t,x) \in \mathbf{R} \times \mathbf{R}^n, n \geq 3.$$

The main property shared by the Cauchy problems (1.1) and (1.2) is that both are invariant under the conformal transformation. Let us recall that the conformal transformation is the map

$$u(t,x) \to \tilde{u}(t,x),$$

defined as follows:

$$\tilde{u}(t,x) = \frac{1}{t^{\frac{n}{2}}} e^{\frac{\mathbf{i}|x|^2}{4t}} u\left(\frac{1}{t}, \frac{x}{t}\right), (t,x) \in (0,\infty) \times \mathbf{R}^n. \tag{1.3}$$

This transformation has been extensively used in the literature in connection with the Schrödinger equation (for more details see [4] and the bibliography therein). In fact an explicit computation shows that if $u(t,x)$ satisfies (1.1) and $\tilde{u}(t,x)$ is defined as in (1.3), then

$$\mathbf{i}\partial_t \tilde{u} + \Delta \tilde{u} - |x|^{-2} W\left(\frac{x}{|x|}\right) \tilde{u} = 0, \tag{1.4}$$

This research was supported by RTN Harmonic Analysis and Related Problems, contract HPRN-CT-2001-00273-HARP. The second author was supported also by the INDAM project "Mathematical Modeling and Numerical Analysis of Quantum Systems with Applications to Nanosciences".

$$\tilde{u}(1) = e^{\mathbf{i}\frac{|x|^2}{4}} u(1), (t,x) \in (0,\infty) \times \mathbf{R}^n, n \geq 3.$$

Similarly if $u(t,x)$ satisfies (1.2), then the corresponding $\tilde{u}(t,x)$ satisfies:

$$\mathbf{i}\partial_t \tilde{u} + \Delta \tilde{u} \mp \tilde{u}|\tilde{u}|^{\frac{4}{n}} = 0, \tag{1.5}$$

$$\tilde{u}(1) = e^{\mathbf{i}\frac{|x|^2}{4}} u(1), (t,x) \in (0,\infty) \times \mathbf{R}^n, n \geq 3.$$

As it has been mentioned above this article is mainly devoted to study the local smoothing for the solutions to the Cauchy problems (1.1) cand (1.2), see [7], [9], [11]. More precisely we shall present some estimates, from above and from below, related with the phenomena of gain of $\frac{1}{2}$ - derivative for the solution to (1.1) and (1.2).

Let us recall also that in the free case (i.e. the linear Schrödinger equation with constant coefficient) these estimates have been already proved in [12]. We were motivated by the results of Agmon and Hörmander in [1]. As it will be clear in the sequel the results that we shall prove will allow us to deduce some new uniqueness criteria for solutions to the Cauchy problems (1.1) and (1.2).

The first result that we shall present concerns the Cauchy problem (1.1).

Theorem 1.1. *Assume that $W : \mathbf{R}^n \to \mathbf{R}$ is bounded, measurable and non - negative and let u be the unique solution to* (1.1) *with initial data $f \in \dot{H}^{\frac{1}{2}}(\mathbf{R}^n)$, then the following a priori estimate is satisfied:*

$$c\|f\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)} \leq \sup_{R>0} \frac{1}{R} \int_0^\infty \int_{|x|<R} |\nabla_x u|^2 \, dxdt \leq C\|f\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)} \tag{1.6}$$

where $c, C > 0$ are suitable constants.

Remark 1.1. As it will be clear in the sequel, the proof of (1.6) will be done by a density argument. Hence we can assume that the initial data f is regular enough in order to guarantee the existence and the uniquess of solution to (1.1).

Remark 1.2. Let us point - out that the r.h.s. estimate in (1.6) has been proved already in the paper [2]. Then our main contribution is the l.h.s. estimate in (1.6). Let us recall also that in [12] the estimate (1.6) has been proved for the free Schrödinger equation, i.e. (1.1) with $W \equiv 0$.

In fact the l.h.s. in (1.6) will follow from the following

Theorem 1.2. *Let W, u and f as in theorem 1.1. Then there exists a constant $c > 0$ such that*

$$\liminf_{R\to\infty} \frac{1}{R} \int_0^\infty \int_{|x|<R} |\nabla_x u|^2 \, dxdt \geq c\|f\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)} \ \forall f \in \dot{H}^{\frac{1}{2}}(\mathbf{R}^n). \tag{1.7}$$

Therefore if

$$\liminf_{R\to\infty} \frac{1}{R} \int_0^\infty \int_{|x|<R} |\nabla_x u|^2 \, dxdt = 0$$

then $u \equiv 0$.

Remark 1.3. Let us recall that the Cauchy problem (1.1) has been investigated in [3] from the point of view of Strichartz estimates, while in [2] it has been studied in connection with the local smoothing phenomena.

Remark 1.4. The non - negativity assumption done on W in theorems 1.1 and 1.2 could be relaxed by assuming a smallness condition on its negative part. However for simplicity we assume $W \geq 0$ in order to avoid technical difficulties and to make more transparent the idea of the proof.

As a by - product of the techniques involved in the proof of the previous theorems, we can get another uniqueness result for solutions to (1.1). As far as we know the content of next result it is not explicitely written elsewhere also in the case of the free Schrödinger equation.

Theorem 1.3. *Let u satisfies (1.1), with $W \geq 0$ and bounded and $f \in L^2(\mathbf{R}^n)$. If you assume that*

$$\liminf_{t\to\infty} \frac{1}{t} \int_{\mathbf{R}^n} |x||u(t,x)|^2 dx = 0,$$

then $u \equiv 0$.

Next we shall present the corresponding version of the previous theorems for the solutions to (1.2).

Theorem 1.4. *Let $\epsilon_0 > 0$ be a small parameter such that (1.2) has a unique global solution $u \in \mathcal{C}(\mathbf{R}; H^1(\mathbf{R}^n) \cap L^{2+\frac{4}{n}}(\mathbf{R} \times \mathbf{R}^n)$, for any f such that $\|f\|_{L^2(\mathbf{R}^n)} < \epsilon_0$. There exists $0 < \epsilon \leq \epsilon_0$ such that if u is the unique solution to (1.2) with initial data f that satisfies $\|f\|_{L^2(\mathbf{R}^n)} < \epsilon$ and moreover*

$$f \in H^1(\mathbf{R}^n) \text{ and } \int_{\mathbf{R}^n} |x|^2 |f(x)|^2 dx < \infty,$$

then the following estimate holds:

$$\begin{aligned} c\|f\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)} &\leq \sup_{R>R_0} \frac{1}{R} \int_0^\infty \int_{|x|<R} |\nabla_x u|^2 \, dxdt \\ &\leq C\left(\|f\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)} + \frac{1}{R_0^{1-\frac{2}{n}}} \|f\|^{\frac{4}{n}}_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)} \right) \quad \forall R_0 > 0 \end{aligned} \tag{1.8}$$

where $c, C > 0$ are universal constants independent of f and $R_0 > 0$. Moreover we have the following chain of inequalities:

$$\begin{aligned} c\|f\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)} &\leq \liminf_{R\to\infty} \frac{1}{R} \int_0^\infty \int_{|x|<R} |\nabla_x u|^2 \, dxdt \\ &\leq \limsup_{R\to\infty} \frac{1}{R} \int_0^\infty \int_{|x|<R} |\nabla_x u|^2 \, dxdt \leq C\|f\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}. \end{aligned} \tag{1.9}$$

Remark 1.5. Let us point - out that the statement of theorem 1.4 contains a global existence result for the Cauchy problem (1.2) under a smallness assumption on the initial data f in $L^2(\mathbf{R}^n)$. Let us recall that this fact has been proved in [5], [6] and [10]. Hence our main contribution concerning the Cauchy problem (1.2) are the estimates (1.8) and (1.9).

We shall prove also the following nonlinear version of theorem 1.3.

Theorem 1.5. *Let u be the unique global solution to (1.2) with $\|f\|_{L^2(\mathbf{R}^n)} < \epsilon_0$, where $\epsilon_0 > 0$ is small enough. Assume moreover that*

$$\liminf_{t\to\infty} \frac{1}{t} \int_{\mathbf{R}^n} |x||u(t,x)|^2 dx = 0,$$

then $u \equiv 0$.

Along the proof of theorem 1.4 (in particular in the proof of the l.h.s. estimate in (1.9)) we shall need some intermediate results that in our opinion have their own interest. One of them will be stated in next theorem.

Theorem 1.6. *Let* $\psi \in C^1(\mathbf{R}^n)$ *be a radially symmetric function such that the following limit exists:*

$$\lim_{|x|\to\infty} \partial_{|x|}\psi(x) = \psi'(\infty) \in [0,\infty), \tag{1.10}$$

and moreover

$$\partial_{|x|}\psi(x) \geq 0 \ \forall x \in \mathbf{R}^n.$$

Assume that u *and* f *are as in theorem 1.4, then the following estimate holds:*

$$\limsup_{t\to\infty}\left(-\mathcal{I}m\int_{\mathbf{R}^n} \bar{u}(t)\nabla u(t)\cdot\nabla\psi(x)\ dx\right) \geq \frac{1}{2}\psi'(\infty)\int_{\mathbf{R}^n}|x||g(x)|^2\ dx, \tag{1.11}$$

where g *is a suitable function that depends on* f *but does not depend on* ψ*. Moreover the following inequality is satisfied:*

$$\|f\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)} \leq C\int_{\mathbf{R}^n}|x||g(x)|^2\ dx, \tag{1.12}$$

where $C > 0$ *is suitable constant that does not depend on* f*.*

Remark 1.6. Looking at the proof of theorem 1.6 it will be clear that the smallness assumption done on the initial condition f in $L^2(\mathbf{R}^n)$ will be relevant in order to prove (1.12). However in the defocusing case (i.e. (1.2) with the sign plus), the existence of the function g, the validity of (1.11) and theorem 1.5 can be proved without any smallness assumption on f and just assuming $f \in H^1(\mathbf{R}^n)$.

Remark 1.7. In the proof of theorems 1.1 and 1.2 we shall need a result similar to theorem 1.6 concerning the solutions to (1.1) (see proposition 3.1). However the proof of theorem 1.6 (and in particular the proof of (1.12)) is much more involved due to the nonlinear nature of the operator $f \to g$ given in the statement of theorem 1.6.

Next we shall fix some notations that will be used in the sequel.

Notations. For any $s \in \mathbf{R}$ we shall denote by $\dot{H}^s_x$ and H^s_x the homogeneous and non - homogeneous Sobolev spaces in $\mathbf{R}^n$ of order s.

For any $R > 0$ we shall denote by B_R the unit ball in $\mathbf{R}^n$ centered in the origin.

For any $1 \leq p, q \leq \infty$

$$L^p_x \text{ and } L^p_t L^q_x$$

denote the Banach spaces

$$L^p(\mathbf{R}^n) \text{ and } L^p(\mathbf{R}; L^q(\mathbf{R}^n)).$$

We shall also write

$$L^p_t L^p_x = L^p_{t,x}.$$

Let X be a general Banach spaces, then $\mathcal{C}_t(X)$ is the space of continuous functions defined in $\mathbf{R}$ and valued in X.

Given any non - negative and measurable function $w : \mathbf{R}^n \to \mathbf{R}^+$ we shall denote by L^2_w the Hilbert space whose norm is defined as follows:

$$\|f\|^2_{L^2_w} = \int_{\mathbf{R}^n} |f(x)|^2 w(x) dx.$$

Given a space - time dependent function $w(t,x)$ we shall denote by $w(t_0)$ the trace of w at fixed time $t \equiv t_0$, in case that it is well - defined.

We shall denote by $\int ...\, dx, \int ...\, dt$ and $\int\int ...\, dxdt$ the integral of suitable functions with respect to the full space, time and space - time variables respectively.

When it is not better specified we shall denote by ∇v the gradient of any time - dependent function $v(t,x)$ with respect to the space variables. Moreover ∇_τ and $\partial_{|x|}$ shall denote respectively the tangential gradient and the radial derivative.

If $\psi \in C^2(\mathbf{R}^n)$, then $D^2\psi$ will represent the hessian matrix of ψ.

2. The conformal conservation law

In order to simplify the proof of our results it will be useful in some cases to work directly with the following general class of Cauchy problems:

$$\mathbf{i}\partial_t u - \Delta u + |x|^{-2} W\left(\frac{x}{|x|}\right) u \pm \lambda u|u|^{\frac{4}{n}} = 0, \tag{2.1}$$

$$u(0) = f, \lambda \geq 0, (t,x) \in \mathbf{R} \times \mathbf{R}^n$$

and

$$\mathbf{i}\partial_t u + \Delta u - |x|^{-2} W\left(\frac{x}{|x|}\right) u \mp \lambda u|u|^{\frac{4}{n}} = 0, \tag{2.2}$$

$$u(0) = f, \lambda \geq 0, (t,x) \in \mathbf{R} \times \mathbf{R}^n.$$

The following result can be found in [4].

Proposition 2.1. *Let $u \in \mathcal{C}_t(H^1_x)$ satisfies (2.1) with $W \geq 0$ and $f \in H^1_x \cap L^2_{|x|^2}$, then:*

$$\int \left(|\nabla u(t)|^2 + |x|^{-2} W\left(\frac{x}{|x|}\right) |u(t)|^2 \pm \frac{\lambda n}{n+2} |u(t)|^{2+\frac{4}{n}} \right) dx \tag{2.3}$$

$$= \int \left(|\nabla f(x)|^2 + |x|^{-2} W\left(\frac{x}{|x|}\right) |f(x)|^2 \pm \frac{\lambda n}{n+2} |f(x)|^{2+\frac{4}{n}} \right) dx \ \forall t \in \mathbf{R};$$

$$\int |u(t)|^2 dx = \int |f(x)|^2 dx \ \forall t \in \mathbf{R}; \tag{2.4}$$

$$\|xu(t) - 2\mathbf{i}t\nabla u(t)\|^2_{L^2_x} + 4t^2 \int |x|^{-2} W\left(\frac{x}{|x|}\right) |u(t)|^2 dx \tag{2.5}$$

$$\pm \frac{2n\lambda t^2}{n+2} \int |u(t)|^{2+\frac{4}{n}} dx = \|f\|^2_{L^2_{|x|^2}} \ \ \forall t \in \mathbf{R}.$$

Notice that if we choose in a suitable way the parameter λ and the potential W in (2.1), then we can deduce from the previous proposition the following corollary.

Corollary 2.1. *Assume that $u \in \mathcal{C}_t(H^1_x)$ solves (1.1) with $W \geq 0$ and $f \in H^1_x \cap L^2_{|x|^2}$, then:*

$$\|xu(t) - 2\mathbf{i}t\nabla u(t)\|^2_{L^2_x} + 4t^2 \int |x|^{-2} W\left(\frac{x}{|x|}\right) |u(t)|^2 dx = \|f\|^2_{L^2_{|x|^2}} \quad \forall t \in \mathbf{R}. \tag{2.6}$$

If $u \in \mathcal{C}_t(H^1_x)$ solves (1.2) with $f \in H^1_x \cap L^2_{|x|^2}$, then:

$$\|xu(t) - 2\mathbf{i}t\nabla u(t)\|^2_{L^2_x} \pm \frac{2nt^2}{n+2} \int |u(t)|^{2+\frac{4}{n}} dx = \|f\|^2_{L^2_{|x|^2}} \quad \forall t \in \mathbf{R}. \tag{2.7}$$

The following proposition is similar to proposition 2.1 except that it concerns the solutions to (2.2).

Proposition 2.2. *Assume that $u \in \mathcal{C}_t(H^1_x)$ satisfies (2.2) where $f \in L^2_x \cap L^2_{|x|^2}$, then the following identities are satisfied:*

$$\int |u(t)|^2 dx = \int |f(x)|^2 dx \quad \forall t \in \mathbf{R}; \tag{2.8}$$

$$\|xu(t) + 2\mathbf{i}t\nabla u(t)\|^2_{L^2_x} + 4t^2 \int |x|^{-2} W\left(\frac{x}{|x|}\right) |u(t)|^2 dx \tag{2.9}$$

$$\pm \frac{2n\lambda t^2}{n+2} \int |u(t)|^{2+\frac{4}{n}} dx = \|f\|^2_{L^2_{|x|^2}} \quad \forall t \in \mathbf{R}.$$

Notice that as a by - product of proposition 2.2 (where we choose $W \equiv 0$ and $\lambda = 0$) we get the following

Corollary 2.2. *Assume that*

$$\mathbf{i}\partial_t u + \Delta u = 0,$$

$$u(0) = f \in L^2_{|x|}$$

then

$$\|e^{-\mathbf{i}\frac{|x|^2}{4t}} u(t)\|^2_{\dot{H}^{\frac{1}{2}}_x} \leq \frac{1}{2t} \|f\|^2_{L^2_{|x|}} \quad \forall t \in (0, \infty).$$

Proof. By using (2.9), where we choose $W \equiv 0$ and $\lambda = 0$, we get:

$$4t^2 \|e^{-\mathbf{i}\frac{|x|^2}{4t}} u(t)\|^2_{\dot{H}^1_x} = 4t^2 \|\nabla(e^{-\mathbf{i}\frac{|x|^2}{4t}} u(t))\|^2_{L^2_x} = \|f\|^2_{L^2_{|x|^2}}.$$

On the other hand the conservation of the charge (see (2.8)) implies

$$\|e^{-\mathbf{i}\frac{|x|^2}{4t}} u(t)\|^2_{L^2_x} = \|u(t)\|_{L^2_x} = \|f\|^2_{L^2_x}.$$

The result follows by interpolation.

$\square$

3. On the asymptotic behaviour of solutions to (1.1) and proof of theorem 1.3

The main result of this section is the following

Proposition 3.1. *Let $\psi \in C^1(\mathbf{R}^n)$ be a radially symmetric function such that the following limit exists:*

$$\lim_{|x|\to\infty} \partial_{|x|}\psi = \psi'(\infty) \in [0,\infty), \tag{3.1}$$

and moreover

$$\partial_{|x|}\psi \geq 0 \ \forall x \in \mathbf{R}^n. \tag{3.2}$$

Let $u \in \mathcal{C}_t(H^1_x)$ be the unique global solution to (1.1)*, where W is bounded and non - negative and $f \in H^1_x \cap L^2_{|x|^2}$, then*

$$\liminf_{t\to\infty} \left(-\mathcal{I}m \int \bar{u}(t)\nabla u(t) \cdot \nabla\psi \ dx \right) \geq \frac{1}{2}\psi'(\infty) \int |x||g(x)|^2 dx, \tag{3.3}$$

where g is a suitable function that depends on f but does not depend on ψ. Moreover the following estimate holds:

$$\|f\|^2_{\dot{H}^{\frac{1}{2}}_x} \leq \frac{1}{2} \int |x||g(x)|^2 dx. \tag{3.4}$$

Proof. Let us notice that (2.6) implies the following identity:

$$\left\| \frac{x}{t}u(t) - 2\mathbf{i}\nabla u(t) \right\|^2_{L^2_x} + 4 \int |x|^{-2} W\left(\frac{x}{|x|}\right) |u(t)|^2 dx = \frac{1}{t^2}\|f\|^2_{L^2_{|x|^2}}, \tag{3.5}$$

that, due to the non - negativity assumption done on W, implies:

$$\lim_{t\to\infty} \left\| \frac{x}{t}u(t) - 2\mathbf{i}\nabla u(t) \right\|^2_{L^2_x} = 0. \tag{3.6}$$

We split now the proof in two parts.

Construction of g and proof of (3.4)

Let us recall that if u satisfies (1.1), then its conformal transformation

$$\tilde{u}(t,x) = \frac{1}{t^{\frac{n}{2}}} e^{\frac{\mathbf{i}|x|^2}{4t}} u\left(\frac{1}{t}, \frac{x}{t}\right),$$

satisfies the Cauchy problem (1.4). In particular

$$\tilde{u}(1) = e^{\frac{\mathbf{i}|x|^2}{4}} u(1) \tag{3.7}$$

and hence

$$\|\tilde{u}(1)\|_{L^2_x} = \|u(1)\|_{L^2_x} = \|f\|_{L^2_x},$$

where we have used the conservation of the charge for the unique solution to (1.1) (see (2.4)). As a consequence $\tilde{u}(t,x)$ satisfies the following Cauchy problem

$$\mathbf{i}\partial_t\tilde{u} + \Delta\tilde{u} - |x|^{-2}W\left(\frac{x}{|x|}\right)\tilde{u} = 0, \tag{3.8}$$

$$\tilde{u}(1) \in L^2_x, (t,x) \in (0,\infty) \times \mathbf{R}^n.$$

Due to the global well - posedness (in the L^2_x sense) of the previous Cauchy problem, we deduce that $\tilde{u}$ can be extended as a solution to the same Cauchy problem in the functional space $\mathcal{C}_t(L^2_x)$. In particular it is well defined a function $g \in L^2_x$ such that the following limit exists in L^2_x:

$$\lim_{t\to 0} \tilde{u}(t,x) = g \in L^2_x. \tag{3.9}$$

Due to (3.9) we can deduce that

$$\lim_{t\to 0} \left\| u\left(\frac{1}{t}, x\right) - t^{\frac{n}{2}} e^{-\mathbf{i}t\frac{|x|^2}{4}} g\,(tx) \right\|_{L^2_x}$$
$$= \lim_{t\to 0} \left\| \frac{1}{t^{\frac{n}{2}}} e^{\mathbf{i}\frac{|x|^2}{4t}} u\left(\frac{1}{t}, \frac{x}{t}\right) - g\,(x) \right\|_{L^2_x} = 0,$$

and in particular

$$\lim_{t\to\infty} \left\| u(t) - \frac{1}{t^{\frac{n}{2}}} e^{-\mathbf{i}\frac{|x|^2}{4t}} g\left(\frac{x}{t}\right) \right\|_{L^2_x} = 0. \tag{3.10}$$

On the other hand (3.8) and (3.9) imply that $\tilde{u}$ satisfies

$$\mathbf{i}\partial_t \tilde{u} + \Delta \tilde{u} - |x|^{-2} W\left(\frac{x}{|x|}\right) \tilde{u} = 0$$
$$\tilde{u}(0) = g$$

that in turn, due to (2.9) (where we choose $\lambda = 0$ and $t = 1$), implies

$$\|x\tilde{u}(1) + 2\mathbf{i}\nabla\tilde{u}(1)\|^2_{L^2_x} + 4\int |x|^{-2} W\left(\frac{x}{|x|}\right) |\tilde{u}(1)|^2 dx = \|g\|^2_{L^2_{|x|^2}}.$$

Notice that this identity is equivalent to

$$4\|\nabla(e^{-\mathbf{i}\frac{|x|^2}{4}}\tilde{u}(1))\|^2_{L^2_x} + 4\int |x|^{-2} W\left(\frac{x}{|x|}\right) |e^{-\mathbf{i}\frac{|x|^2}{4}}\tilde{u}(1)|^2 dx = \|g\|^2_{L^2_{|x|^2}},$$

that due to (3.7) gives:

$$4\|\nabla u(1)\|^2_{L^2_x} + 4\int |x|^{-2} W\left(\frac{x}{|x|}\right) |u(1)|^2 dx = \|g\|^2_{L^2_{|x|^2}}. \tag{3.11}$$

By combining this identity with (2.3) (where we choose $\lambda = 0$) and with the non - negativity assumption done on W we get:

$$4\|\nabla f\|^2_{L^2_x} \le 4\|\nabla f\|^2_{L^2_x} + 4\int |x|^{-2} W\left(\frac{x}{|x|}\right) |f(x)|^2 dx \tag{3.12}$$
$$= 4\|\nabla u(1)\|^2_{L^2_x} + 4\int |x|^{-2} W\left(\frac{x}{|x|}\right) |u(1)|^2 dx = \|g\|^2_{L^2_{|x|^2}}.$$

Notice also that the following estimate follows easily from the conservation of the charge (see (2.4) and (2.8)):

$$\|f\|^2_{L^2_x} = \|u(1)\|^2_{L^2_x} = \|\tilde{u}(1)\|^2_{L^2_x} = \|g\|^2_{L^2_x}. \tag{3.13}$$

Then (3.4) follows by making interpolation between (3.12) and (3.13).

Proof of (3.3)

Due to (3.6) we can deduce that

$$\lim_{t\to\infty}\left[\int \bar{u}(t)\nabla u(t)\cdot\nabla\psi\ dx+\frac{\mathbf{i}}{2t}\int |x||u(t)|^2\partial_{|x|}\psi\ dx\right]=0 \tag{3.14}$$

and then

$$\begin{aligned}\liminf_{t\to\infty}&\left(-\mathcal{I}m\int \bar{u}(t)\nabla u(t)\cdot\nabla\psi dx\right)\\ &=\frac{1}{2}\liminf_{t\to\infty}\int |x||u(t)|^2\partial_{|x|}\psi\ \frac{dx}{t}.\end{aligned} \tag{3.15}$$

Next we fix a real number $R>0$ and we notice that due to the non - negativity assumption done on $\partial_{|x|}\psi$ (see (3.2)) we get:

$$\begin{aligned}\int |x||u(t)|^2\partial_{|x|}\psi\ \frac{dx}{t}&\geq\int_{|x|\leq Rt}|x||u(t)|^2\partial_{|x|}\psi\ \frac{dx}{t}\\ =\int_{|x|\leq Rt}&|x|\left(|u(t)|^2-\frac{1}{t^n}\left|g\left(\frac{x}{t}\right)\right|^2\right)\partial_{|x|}\psi\ \frac{dx}{t}\\ +\int_{|x|\leq Rt}&|x|\left((\partial_{|x|}\psi-\psi'(\infty)\right)\left|g\left(\frac{x}{t}\right)\right|^2\frac{dx}{t^{n+1}}\\ +\psi'(\infty)&\int_{|x|\leq Rt}|x|\left|g\left(\frac{x}{t}\right)\right|^2\frac{dx}{t^{n+1}}\ \forall R>0,\end{aligned} \tag{3.16}$$

where g is the function constructed in the previous step.

Notice that the following estimate is trivial:

$$\begin{aligned}&\int_{|x|\leq Rt}|x|\left(|u(t)|^2-\frac{1}{t^n}\left|g\left(\frac{x}{t}\right)\right|^2\right)\partial_{|x|}\psi\ \frac{dx}{t}\\ \leq R\|\partial_{|x|}\psi\|_{L^\infty_x}&\int_{|x|\leq Rt}\left||u(t)|^2-\frac{1}{t^n}\left|g\left(\frac{x}{t}\right)\right|^2\right|dx\to 0\text{ as }t\to\infty,\end{aligned} \tag{3.17}$$

where at the last step we have used (3.10).

Moreover the change of variable formula implies:

$$\begin{aligned}&\left|\int_{|x|\leq Rt}|x|\left((\partial_{|x|}\psi-\psi'(\infty)\right)\left|g\left(\frac{x}{t}\right)\right|^2\frac{dx}{t^{n+1}}\right|\\ &\leq R\int_{|x|\leq R}\left|\partial_{|x|}\psi(tx)-\psi'(\infty)\right||g(x)|^2dx,\end{aligned}$$

that in conjunction with the dominated convergence theorem and with assumption (3.1) implies:

$$\lim_{t\to\infty}\int_{|x|\leq Rt}|x|\left((\partial_{|x|}\psi-\psi'(\infty)\right)\left|g\left(\frac{x}{t}\right)\right|^2\frac{dx}{t^{n+1}}=0\ \forall R>0. \tag{3.18}$$

Due again to the change of variable formula we get

$$\psi'(\infty)\int_{|x|\leq Rt}|x|\left|g\left(\frac{x}{t}\right)\right|^2\frac{dx}{t^{n+1}}=\psi'(\infty)\int_{|x|\leq R}|x||g(x)|^2dx,$$

and in particular

$$\lim_{t\to\infty}\psi'(\infty)\int_{|x|\leq Rt}|x|\left|g\left(\frac{x}{t}\right)\right|^2\frac{dx}{t^{n+1}}=\psi'(\infty)\int_{|x|\leq R}|x||g(x)|^2dx. \tag{3.19}$$

By combining (3.17),(3.18), (3.19) and (3.16) we can deduce that

$$\liminf_{t\to\infty} \int |x||u(t)|^2 \partial_{|x|}\psi(x)\frac{dx}{t} \tag{3.20}$$

$$\geq \psi'(\infty)\int_{|x|\leq R} |x||g(x)|^2 dx \;\forall R>0.$$

Since $R>0$ is arbitrary, we can combine (3.15) with (3.20) in order to deduce (3.3).

□

Proof of theorem 1.3 Let g be the function constructed in proposition 3.1. Looking at the proof of (3.20) it is easy to deduce with a similar argument the following estimate:

$$\liminf_{t\to\infty} \int |x||u(t)|^2 \frac{dx}{t} \geq \int_{|x|\leq R} |x||g(x)|^2 dx \;\forall R>0. \tag{3.21}$$

In fact we have:

$$\int |x||u(t)|^2 \frac{dx}{t} \geq \int_{|x|\leq Rt} |x||u(t)|^2 \;\frac{dx}{t} \tag{3.22}$$

$$= \int_{|x|\leq Rt} |x|\left[|u(t)|^2 - \frac{1}{t^n}\left|g\left(\frac{x}{t}\right)\right|^2\right]\frac{dx}{t} + \int_{|x|\leq Rt} |x|\left|g\left(\frac{x}{t}\right)\right|^2 \frac{dx}{t^{n+1}} \;\forall R>0.$$

Notice that the following estimate is trivial:

$$\int_{|x|\leq Rt} |x|\left(|u(t)|^2 - \frac{1}{t^n}\left|g\left(\frac{x}{t}\right)\right|^2\right)\frac{dx}{t} \tag{3.23}$$

$$\leq R\int_{|x|\leq Rt} \left||u(t)|^2 - \frac{1}{t^n}\left|g\left(\frac{x}{t}\right)\right|^2\right| dx \to 0 \text{ as } t\to\infty,$$

where at the last step we have used (3.10).

On the other hand we have:

$$\int_{|x|\leq Rt} |x|\left|g\left(\frac{x}{t}\right)\right|^2 \frac{dx}{t^{n+1}} = \int_{|x|\leq R} |x||g(x)|^2 dx. \tag{3.24}$$

and then (3.21) follows by combining (3.22), (3.23), (3.24).

In particular if

$$\liminf_{t\to\infty} \int |x||u(t)|^2 \frac{dx}{t} = 0,$$

then (3.21) implies $g\equiv 0$, that in turn due to (3.10) gives $\lim_{t\to\infty}\|u(t)\|_{L^2_x} = 0$. By combining this fact with the conservation of the charge (2.4), we get $f\equiv 0$, and hence $u\equiv 0$.

□

4. Proof of theorems 1.1 and 1.2

In the first part of this section we recall the approach used in [2] in order to deduce the local smoothing estimate (i.e. the r.h.s. in (1.6)) for the solutions to (1.1). The main idea is to multiply (1.1) by the quantity

$$\nabla\bar{u}\cdot\nabla\psi + \frac{1}{2}\bar{u}\;\Delta\psi, \tag{4.1}$$

and to integrate on the strip $(0,T)\times \mathbf{R}^n$. For the moment $\psi:\mathbf{R}^n\to\mathbf{R}$ is a general function to which we require only minimal regularity assumptions in order to justify the integration by parts.

The approach described above allows you to deduce the following family of identities:

$$\int_0^T\int\left[\nabla\bar{u}D^2\psi\nabla u-\frac{1}{4}|u|^2\Delta^2\psi+|u|^2|x|^{-3}W\left(\frac{x}{|x|}\right)\partial_{|x|}\psi\right]dxdt \tag{4.2}$$
$$=-\frac{1}{2}\mathcal{I}m\int\bar{u}(T)\nabla u(T)\cdot\nabla\psi\ dx+\frac{1}{2}\mathcal{I}m\int\bar{f}\ \nabla f\cdot\nabla\psi\ dx,$$

(for more details on this computation see [2] and [12]).

The following propositions will be relevant in the sequel.

Proposition 4.1. *Let $\psi\in C^1(\mathbf{R}^n)$ be a radially symmetric function such that:*

$$|\partial_{|x|}\psi|,|x||\partial^2_{|x|}\psi|\le C<\infty\ \forall x\in\mathbf{R}^n,$$

where $C>0$ is a suitable constant. Then there exists $C'>0$, that depends only on C and such that:

$$\left|\int g(x)\nabla\bar{g}(x)\cdot\nabla\psi\ dx\right|\le C'\|g\|^2_{\dot{H}^{\frac{1}{2}}_x}\ \forall g\in H^1_x. \tag{4.3}$$

In particular if $u\in\mathcal{C}_t(H^1_x)$ is the unique solution to the Cauchy problem (1.1) *where $W\ge 0$ and bounded and $f\in H^1_x$, then:*

$$\left|\int u(t)\nabla\bar{u}(t)\cdot\nabla\psi\ dx\right|\le C'\|f\|^2_{\dot{H}^{\frac{1}{2}}_x}\ \forall t\in\mathbf{R}. \tag{4.4}$$

Proof. The proof of (4.3) can be found in [2]. In order to prove (4.4) let us notice that if we choose $g(x)=u(t)$ in (4.3) then the inequality becomes

$$\left|\int u(t)\nabla\bar{u}(t)\cdot\nabla\psi\ dx\right|\le C'\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}_x}. \tag{4.5}$$

On the other hand due to (2.3) and by recalling the non - negativity assumption done on W we get:

$$\|u(t)\|^2_{\dot{H}^1_x}\le\|f\|^2_{\dot{H}^1_x}+\|W\|_{L^\infty_x}\int|x|^{-2}|f(x)|^2dx\le C\|f\|^2_{\dot{H}^1_x},$$

where we have used the classical Hardy inequality at the last step.

By making interpolation between this inequality and

$$\|u(t)\|^2_{L^2_x}=\|f\|^2_{L^2_x}$$

that follows from (2.4), we deduce

$$\|u(t)\|_{\dot{H}^{\frac{1}{2}}_x}\le C\|f\|_{\dot{H}^{\frac{1}{2}}_x}\ \forall t\in\mathbf{R}.$$

By combining this last inequality with (4.5) we get (4.4). □

Corollary 4.1. *Assume that $u\in\mathcal{C}_t(H^1_x)$ solves* (1.1) *with $W\ge 0$ and bounded and $f\in H^1_x$, then*

$$\int\int_{|x|<1}\frac{|u|^2}{|x|^2}W\left(\frac{x}{|x|}\right)dxdt<\infty. \tag{4.6}$$

Proof. Let us notice that if we choose in (4.2) the function ψ to be equal to the function ϕ given in proposition 7.1 (see the Appendix) then we get the following inequality:

$$\int\int_{|x|<1} |u|^2|x|^{-3}W\left(\frac{x}{|x|}\right)\partial_{|x|}\phi \, dxdt \leq C\|f\|^2_{\dot{H}^{\frac{1}{2}}_x}, \tag{4.7}$$

where we have used also (4.4).

On the other hand by the Taylor formula and by using the properties of ϕ we get

$$\partial_{|x|}\phi(|x|) = \partial^2_{|x|}\phi(0)|x| + 0(|x|^2).$$

By combining this identity with (4.7) we get

$$\partial^2_{|x|}\phi(0)\int\int_{|x|<1} |u|^2|x|^{-2}W\left(\frac{x}{|x|}\right) dxdt \tag{4.8}$$

$$\leq C\|f\|^2_{\dot{H}^{\frac{1}{2}}_x} + C\int\int_{|x|<1} |u|^2|x|^{-1}W\left(\frac{x}{|x|}\right) dxdt.$$

On the other hand the Hölder inequality implies:

$$\int\int_{|x|<1} |u|^2|x|^{-1}W\left(\frac{x}{|x|}\right) dxdt \tag{4.9}$$

$$\leq \|W\|_{L^\infty_x}\|u\|^2_{L^2_t L^{\frac{2n}{n-2}}_x}\||x|^{-1}\|_{L^{\frac{n}{2}}(|x|<1)} < \infty$$

where we have used in the last step the fact that Strichartz estimates are satisfied by the solutions to (1.1) (see [3]).

Since $\partial^2_{|x|}\phi(0) > 0$ by construction, we can combine (4.8) with (4.9) in order to get the desired result.

□

Remark 4.1. Notice that following [2] it is possible to show that

$$\int\int_{|x|<1} \frac{|u|^2}{|x|^{3-\epsilon}} \, dxdt < \infty$$

for any $\epsilon > 0$ and for any u that satisfies (1.1). However in order to make this paper self - contained we have presented a simplified argument to prove corollary 4.1 that is enough for our purpose.

Next proposition follows by combining proposition 3.1 with (4.2).

Proposition 4.2. *Assume that ψ satisfies the same assumptions as in proposition 3.1 and u satisfies* (1.1) *with $W \geq 0$ and bounded and $f \in H^1_x \cap L^2_{|x|^2}$. Then the following estimate holds:*

$$\int_0^T\int\left[\nabla\bar{u}D^2\psi\nabla u - \frac{1}{4}|u|^2\Delta^2\psi + |u|^2|x|^{-3}W\left(\frac{x}{|x|}\right)\partial_{|x|}\psi\right] dxdt \tag{4.10}$$

$$\geq \frac{1}{4}\psi'(\infty)\|f\|^2_{\dot{H}^{\frac{1}{2}}_x} + \frac{1}{2}\mathcal{I}m\int \bar{f}\,\nabla f\cdot\nabla\psi \, dx.$$

We shall need also the following

Lemma 4.1. *Assume that $u \in \mathcal{C}_t(H^1_x)$ satisfies* (1.1) *with $f \in H^1_x$ and $W \geq 0$ and bounded, then:*

$$\lim_{R\to\infty} \int\int |u|^2 |\Delta^2 \phi_R| \ dxdt = 0, \tag{4.11}$$

$$\lim_{R\to\infty} \int\int \frac{|u|^2}{|x|^3} W\left(\frac{x}{|x|}\right) |\partial_{|x|}\phi_R| \ dxdt = 0 \tag{4.12}$$

where $\phi \in C^4(\mathbf{R}^n)$ is a radially symmetric function such that

$$\partial_{|x|}\phi(0) = 0,$$

$$|\partial_{|x|}\phi| \leq C, |\Delta^2\phi| \leq \frac{C}{(1+|x|)^3} \ \forall x \in \mathbf{R}^n$$

and $\phi_R = R\phi\left(\frac{x}{R}\right)$.

Proof.
Proof of (4.11)

Notice that the Hölder inequality implies

$$\int\int |u|^2|\Delta^2\phi_R| \ dxdt \leq \left[\int \left(\int |u(t)|^{\frac{2n}{n-2}} \ dx\right)^{\frac{n-2}{n}} dt\right] \left(\int |\Delta^2 \phi_R|^{\frac{n}{2}} \ dx\right)^{\frac{2}{n}}$$

$$\leq C\|u\|^2_{L^2_t L^{\frac{2n}{n-2}}_x} \left(\int \frac{1}{(R+|x|)^{\frac{3n}{2}}} \ dx\right)^{\frac{2}{n}} \to 0 \text{ as } R \to \infty,$$

where we have used the estimate $\|u\|_{L^2_t L^{\frac{2n}{n-2}}_x} < \infty$ (whose proof can be found in [3]).

Proof of (4.12)

By using the Hölder inequality we get:

$$\left| \int\int_{|x|>1} \frac{|u|^2}{|x|^3} W\left(\frac{x}{|x|}\right) |\partial_{|x|}\phi_R| \ dxdt \right| \tag{4.13}$$

$$\leq \|W\|_{L^\infty_x} \left[\int (\int |u(t)|^{\frac{2n}{n-2}} dx)^{\frac{n-2}{n}} dt\right] \left(\int_{|x|>1} |x|^{-\frac{3n}{2}} |\partial_{|x|}\phi_R|^{\frac{n}{2}} dx\right)^{\frac{2}{n}}.$$

Notice also that

$$\int_{|x|>1} |x|^{-\frac{3n}{2}} |\partial_{|x|}\phi_R|^{\frac{n}{2}} dx$$

$$= \int_{1<|x|<R} |x|^{-\frac{3n}{2}} |\partial_{|x|}\phi_R|^{\frac{n}{2}} dx + \int_{|x|>R} |x|^{-\frac{3n}{2}} |\partial_{|x|}\phi_R|^{\frac{n}{2}} dx$$

$$\leq \frac{1}{R^{\frac{n}{2}}} \int_{1<|x|<R} \frac{1}{|x|^n} \ dx + C \int_{|x|>R} |x|^{-\frac{3n}{2}} dx$$

where we have used

$$\partial_{|x|}\phi\left(\frac{x}{R}\right) = 0\left(\frac{|x|}{R}\right).$$

By combining this estimate with (4.13) and recalling that $\|u\|_{L^2_t L_x^{\frac{2n}{n-2}}} < \infty$, we get

$$\left| \int \int_{|x|>1} \frac{|u|^2}{|x|^3} W\left(\frac{x}{|x|}\right) |\partial_{|x|}\phi_R| \ dxdt \right| \to 0 \text{ as } R \to \infty.$$

Next we treat the integral in the cylinder $\{|x| < 1\} \times \mathbf{R}$. We use the Taylor formula as above and we get:

$$\partial_{|x|}\phi(|x|) = 0(|x|) \text{ as } |x| \to 0$$

and then

$$\left| \int \int_{|x|<1} \frac{|u|^2}{|x|^3} W\left(\frac{x}{|x|}\right) |\partial_{|x|}\phi_R| \ dxdt \right|$$

$$\leq \frac{C}{R} \left| \int \int_{|x|<1} \frac{|u|^2}{|x|^2} W\left(\frac{x}{|x|}\right) dxdt \right| \to 0 \text{ as } R \to \infty,$$

where at the last step we have used (4.6).

□

We are now able to prove theorems 1.1 and 1.2.

Proof of theorems 1.1 and 1.2.

Due to a density argument it is sufficient to prove the theorems for $f \in H^1_x \cap L^2_{|x|^2}$.

Next we split the proof in two steps.

Proof of r.h.s. in (1.6)

It is sufficient to replace in (4.2) the generic function ψ with the family of rescaled functions $R\phi\left(\frac{x}{R}\right)$, where ϕ is a function that satisfies proposition 7.1 and by estimating the r.h.s. in (4.2) by using (4.4).

Let us point - out that the l.h.s. in (1.6) follows from theorem 1.2.

Proof of theorem 1.2

First of all let us notice that if we choose in the identity (4.2) the function ψ to be equal to the function ϕ given in proposition 7.1, then it is not difficult to verify that

$$\int_{|x|>1} \frac{|\nabla_\tau u|^2}{|x|} \ dx < \infty,$$

provided that $f \in H^1_x$, and in particular

$$\lim_{R\to\infty} \int_{|x|>R} \frac{|\nabla_\tau u|^2}{|x|} \ dx = 0. \tag{4.14}$$

Let us fix a function $h(r) \in C^\infty_0(\mathbf{R}; [0,1])$ such that:

$$h(r) \equiv 1 \ \forall r \in \mathbf{R} \text{ s.t. } |r| < 1, h(r) \equiv 0 \ \forall r \in \mathbf{R} \text{ s.t. } |r| > 2,$$

$$h(r) = h(-r) \ \forall r \in \mathbf{R}.$$

We introduce also the functions $\phi, H \in C^\infty(\mathbf{R})$:

$$\phi(r) = \int_0^r (r-s)h(s)ds \ \text{ and } \ H(r) = \int_0^r h(s)ds,$$

(since now on in the proof the function ϕ will be the one defined above and not the one given in proposition 7.1). Notice that

$$\phi''(r) = h(r), \phi'(r) = H(r) \ \forall r \in \mathbf{R} \text{ and } \lim_{r\to\infty} \partial_{|x|}\phi(r) = \int_0^\infty h(s)ds. \tag{4.15}$$

Moreover an elementary computation shows that:

$$\Delta^2\phi = \frac{C}{|x|^3} \ \forall x \in \mathbf{R}^n \text{ s.t. } |x| \geq 2, \tag{4.16}$$

where Δ^2 is the bilaplacian operator.

Thus the function ϕ defined above satisfies the assumptions of lemma 4.1. Notice also that the assumptions in proposition 3.1 are satisfied by ϕ.

In the sequel we shall need the rescaled functions

$$\phi_R = R\phi\left(\frac{x}{R}\right) \ \forall x \in \mathbf{R}^n \text{ and } R > 0,$$

(where ϕ is the function defined above) and we shall exploit the following elementary identity:

$$\nabla\bar{u}D^2\psi\nabla u = \partial^2_{|x|}\psi|\partial_{|x|}u|^2 + \frac{\partial_{|x|}\psi}{|x|}|\nabla_\tau u|^2, \tag{4.17}$$

where ψ is any regular radial function and u is another regular function.

By combining this identity with (4.2) and with proposition 4.2, where we choose $\psi = \phi_R$, and recalling (4.15) we get:

$$\begin{aligned}\int_0^\infty \int \Big[\partial^2_{|x|}\phi_R|\partial_{|x|}u|^2 + \frac{\partial_{|x|}\phi_R}{|x|}|\nabla_\tau u|^2 \\ -\frac{1}{4}|u|^2\Delta^2\phi_R + \frac{|u|^2}{|x|^3}W\left(\frac{x}{|x|}\right)\partial_{|x|}\phi_R\Big]\, dxdt \\ \geq \frac{1}{4}\left(\int_0^\infty h(s)ds\right)\|f\|^2_{\dot{H}^{\frac{1}{2}}_x} + \frac{1}{2}\mathcal{I}m\int \bar{f}\ \nabla f \cdot \nabla\phi\left(\frac{x}{R}\right)dx \ \forall R > 0.\end{aligned} \tag{4.18}$$

By using now (4.11) and (4.12) we get

$$\lim_{R\to\infty}\int_0^\infty \int \left[-\frac{1}{4}|u(t)|^2\Delta^2\phi_R + \frac{|u(t)|^2}{|x|^3}W\left(\frac{x}{|x|}\right)|\partial_{|x|}\phi_R|\right]dxdt = 0.$$

On the other hand since ϕ is a radially symmetric function we have

$$\lim_{x\to 0}\nabla\phi(x) = 0,$$

that due to the dominated convergence theorem implies

$$\lim_{R\to 0}\int \bar{f}\ \nabla f \cdot \nabla\phi\left(\frac{x}{R}\right)dx = 0.$$

By combining these facts with (4.18) we get

$$\begin{aligned}\liminf_{R\to\infty}\int_0^\infty \int \left(\partial^2_{|x|}\phi_R|\partial_{|x|}u|^2 + \frac{\partial_{|x|}\phi_R}{|x|}|\nabla_\tau u|^2\right)dxdt \\ \geq \frac{1}{4}\left(\int_0^\infty h(s)ds\right)\|f\|^2_{\dot{H}^{\frac{1}{2}}_x}.\end{aligned} \tag{4.19}$$

On the other hand by using the cut - off properties of h, (4.14) and noticing that $\partial_{|x|}\phi = 0(|x|)$ as $|x| \to 0$, we get:

$$\liminf_{R\to\infty} \frac{1}{R}\int_0^\infty \int_{B_{2R}} (|\partial_{|x|}u|^2 + |\nabla_\tau u|^2)\, dxdt \tag{4.20}$$
$$\geq \liminf_{R\to\infty} \int_0^\infty \int \left(\partial^2_{|x|}\phi_R |\partial_{|x|}u|^2 + \frac{\partial_{|x|}\phi_R}{|x|}|\nabla_\tau u|^2\right) dxdt$$
$$\geq \frac{1}{4}\left(\int_0^\infty h(s)ds\right) \|f\|^2_{\dot{H}^{\frac{1}{2}}_x}$$

where we have used (4.19) at the last step. The proof is complete.

□

5. On the asymptotic behaviour of solutions to critical NLS and proof of theorems 1.5, 1.6

The main aim of this section is to prove theorem 1.6 that represents the nonlinear version of proposition 3.1.

First of all let us recall a precise statement about the global existence result to (1.2) with small initial data.

Theorem 5.1. *There exists $\epsilon_0 > 0$ such that for any $f \in L^2_x$ with $\|f\|_{L^2_x} < \epsilon_0$, the Cauchy problem* (1.2) *has an unique global solution*

$$u \in \mathcal{C}_t(L^2_x) \cap L^{2+\frac{4}{n}}_{t,x}.$$

Moreover there exists a function

$$(0, \epsilon_0) \ni \epsilon \to R(\epsilon) \in \mathbf{R}^+$$

such that:

(1) $\lim_{\epsilon\to 0} R(\epsilon) = 0$
(2) *if $\|f\|_{L^2_x} < \epsilon$ then then unique solution to* (1.2) *satisfies* $\|u\|_{L^{2+\frac{4}{n}}_{t,x}} < R(\epsilon)$.

If moreover we assume $f \in H^1_x$, then $u \in \mathcal{C}_t(H^1_x)$.

Remark 5.1. Notice that theorem 5.1 provides a global existence result for small initial data, and also the arbitrary smallness of the $L^{2+\frac{4}{n}}_{t,x}$ - norm of the solutions, provided that the initial data are small enough. Let us recall that in the defocusing case it is sufficient to assume $f \in H^1_x$ to have a global solution and no extra smallness assumption is needed.

In the sequel we shall make extensively use of the following inequality:

$$\||D|^s(u|u|^q)\|_{L^p_x} \leq C\||D|^s u\|_{L^r_x}\|u\|^q_{L^{sq}_x} \tag{5.1}$$

where $C = C(s, q, p, r, s) > 0$, $0 \leq s \leq 1$, $1 < p, r, s < \infty$ and

$$\frac{1}{p} = \frac{1}{r} + \frac{1}{s}.$$

We shall also make use of the classical Strichartz inequalities that we mention below for completeness (for a proof see [8]).

Assume that u solves the following Schrödinger equation with forcing term:

$$\mathbf{i}\partial_t u \pm \Delta u = F(t, x)$$
$$u(0) = f, (t, x) \in \mathbf{R}_t \times \mathbf{R}^n_x, n \geq 3$$

then the following a - priori estimates are satisfied:

$$\|u\|_{L_t^p L_x^q} \leq C\left(\|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}\right),$$

where

$$\frac{2}{p} + \frac{n}{q} = \frac{2}{\tilde{p}} + \frac{n}{\tilde{q}} = \frac{n}{2},\ 2 \leq p, \tilde{p} \leq \infty,$$
$$\frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1,$$

and $C = C(p, q, \tilde{p}, \tilde{q}) > 0$.

Next we shall prove some preliminary propositions that will be useful along the proof of theorem 1.6.

Proposition 5.1. *Let $v \in \mathcal{C}_t(H_x^1) \cap L_{t,x}^{2+\frac{4}{n}}$ be the unique solution to*

$$\mathbf{i}\partial_t v + \Delta v \mp v|v|^{\frac{4}{n}} = 0,$$
$$v(0) = g$$

where $g \in L_{|x|}^2 \cap H_x^1$ and $\|g\|_{L_x^2} < \epsilon$, with $\epsilon > 0$ small enough. Then the following estimate holds:

$$\|e^{-\mathbf{i}\frac{|x|^2}{4}} v(1)\|_{\dot{H}_x^{\frac{1}{2}}} \leq C\|g\|_{L_{|x|}^2}. \tag{5.2}$$

Proof. Let us introduce the function

$$v^*(t,x) = \frac{1}{t^{\frac{n}{2}}} e^{-\mathbf{i}\frac{|x|^2}{4t}} v\left(\frac{x}{t}, \frac{1}{t}\right).$$

It is easy to verify that v^* satisfies:

$$\mathbf{i}\partial_t v^* - \Delta v^* \pm v^*|v^*|^{\frac{4}{n}} = 0, \tag{5.3}$$
$$v^*(1) = e^{-\mathbf{i}\frac{|x|^2}{4}} v(1)$$

and then

$$\mathbf{i}\partial_t(|D|^s v^*) - \Delta(|D|^s v^*) \pm |D|^s(v^*|v^*|^{\frac{4}{n}}) = 0, \tag{5.4}$$
$$(|D|^s v^*)(1) = |D|^s(e^{-\mathbf{i}\frac{|x|^2}{4}} v(1)).$$

Hence $|D|^s v^*$ satisfies the free Schrödinger equation with forcing term given by $\mp|D|^s(v^*|v^*|^{\frac{4}{n}})$.

We can then apply Strichartz estimates for a suitable choice of the parameters $p, q, \tilde{p}, \tilde{q}$ in order to get:

$$\||D|^s v^*\|_{L_t^2 L_x^{\frac{2n}{n-2}}} \leq C\left(\||D|^s v^*(1)\|_{L_x^2} + \||D|^s(v^*|v^*|^{\frac{4}{n}})\|_{L_t^{\frac{2n+4}{n+6}} L_x^{\frac{2n(n+2)}{n^2-4+4n}}}\right)$$
$$\leq C\left(\||D|^s v^*(1)\|_{L_x^2} + \||D|^s v^*\|_{L_t^2 L_x^{\frac{2n}{n-2}}} \|v^*\|_{L_{t,x}^{2+\frac{4}{n}}}^{\frac{4}{n}}\right)$$

where at the last step we have used (5.1). On the other hand $\|v^*\|_{L_{t,x}^{2+\frac{4}{n}}}$ is small (this fact follows from the smallness of v that in turn follows from the smallness assumption done on g, see remark 5.1) and then we get from the previous estimate the following one:

$$\||D|^s v^*\|_{L_t^2 L_x^{\frac{2n}{n-2}}} \leq C\||D|^s v^*(1)\|_{L_x^2}. \tag{5.5}$$

Let us introduce now the functions $w(t,x)$ and $z(t,x)$ defined as the unique solutions to:

$$\mathbf{i}\partial_t z + \Delta z = 0, \tag{5.6}$$
$$z(0) = g$$

and

$$\mathbf{i}\partial_t w + \Delta w \mp v|v|^{\frac{4}{n}} = 0, \tag{5.7}$$
$$w(0) = 0.$$

It is clear that the following identity holds:

$$z + w = v. \tag{5.8}$$

Along with z and w we introduce also

$$w^* = \frac{1}{t^{\frac{n}{2}}} e^{-\mathbf{i}\frac{|x|^2}{4t}} w\left(\frac{1}{t}, \frac{x}{t}\right)$$

and

$$z^* = \frac{1}{t^{\frac{n}{2}}} e^{-\mathbf{i}\frac{|x|^2}{4t}} z\left(\frac{1}{t}, \frac{x}{t}\right).$$

Notice that due to (5.8) we have also

$$v^* = z^* + w^*. \tag{5.9}$$

Next we shall estimate separately z and w.

Estimate for z

Since z is defined by (5.6), we can apply corollary 2.2 for $t = 1$, in order to deduce

$$\||D|^{\frac{1}{2}} z^*(1)\|_{L^2_x} = \||D|^{\frac{1}{2}} [e^{-\mathbf{i}\frac{|x|^2}{4}} z(1)]\|_{L^2_x} \leq \frac{1}{\sqrt{2}} \|g\|_{L^2_{|x|}}. \tag{5.10}$$

Estimate for w

Let us notice that the following identity trivially holds:

$$\|w^*(t)\|_{L^2_x} = \left\| w\left(\frac{1}{t}\right) \right\|_{L^2_x}.$$

On the other hand (by definition) $w(0) = 0$ and then due to the previous identity we get

$$\lim_{t\to\infty} \|w^*(t)\|_{L^2_x} = 0. \tag{5.11}$$

Moreover the function w^* satisfies:

$$\mathbf{i}\partial_t w^* - \Delta w^* = \mp v^*|v^*|^{\frac{4}{n}}, \tag{5.12}$$
$$w^*(1) = e^{-\mathbf{i}\frac{|x|^2}{4}} w(1)$$

and then

$$\mathbf{i}\partial_t(|D|w^*) - \Delta(|D|w^*) = \mp |D|(v^*|v^*|^{\frac{4}{n}}), \tag{5.13}$$
$$|D|w^*(1) = |D|(e^{-\mathbf{i}\frac{|x|^2}{4}} w(1)).$$

We can combine again as above the Strichartz estimates with (5.1) in order to deduce:

$$\||D|w^*\|_{L^\infty_t L^2_x} \leq C\left(\||D|w^*(1)\|_{L^2_x} + \||D|(v^*|v^*|^{\frac{4}{n}})\|_{L_t^{\frac{2n+4}{n+6}} L_x^{\frac{2n(n+2)}{n^2-4+4n}}}\right) \tag{5.14}$$

$$\leq C\left(\||D|w^*(1)\|_{L^2_x} + \||D|v^*\|_{L^2_t L_x^{\frac{2n}{n-2}}} \|v^*\|^{\frac{4}{n}}_{L_{t,x}^{2+\frac{4}{n}}}\right)$$

$$\leq C\left(\||D|w^*(1)\|_{L^2_x} + \||D|v^*(1)\|_{L^2_x}\|v^*\|^{\frac{4}{n}}_{L_{t,x}^{2+\frac{4}{n}}}\right),$$

where we have used (5.5) for $s = 1$ at the last step.

In particular we have

$$\sup_{t\in\mathbf{R}} \||D|w^*(t)\|_{L^2_x} < \infty.$$

By combining this fact with (5.11) and with the following elementary estimate:

$$\||D|^{\frac{1}{2}}\phi\|_{L^2_x} \leq C\|\phi\|^{\frac{1}{2}}_{L^2_x}\||D|\phi\|^{\frac{1}{2}}_{L^2_x} \ \forall \phi \in H^1_x,$$

we deduce that

$$\lim_{t\to\infty} \||D|^{\frac{1}{2}}w^*(t)\|_{L^2_x} = 0.$$

As a consequence of this fact and (5.12) we have that

$$\mathbf{i}\partial_t(|D|^{\frac{1}{2}}w^*) - \Delta(|D|^{\frac{1}{2}}w^*) = \mp|D|^{\frac{1}{2}}(v^*|v^*|^{\frac{4}{n}}), \tag{5.15}$$

$$(|D|^{\frac{1}{2}}w^*)(\infty) = 0.$$

By combining again Strichartz estimates (with the initial condition at infinity) with (5.1) we get:

$$\||D|^{\frac{1}{2}}w^*\|_{L^\infty_t L^2_x} \leq C\||D|^{\frac{1}{2}}(v^*|v^*|^{\frac{4}{n}})\|_{L_t^{\frac{2n+4}{n+6}} L_x^{\frac{2n(n+2)}{n^2-4+4n}}} \tag{5.16}$$

$$\leq C\||D|^{\frac{1}{2}}v^*\|_{L^2_t L_x^{\frac{2n}{n-2}}} \|v^*\|^{\frac{4}{n}}_{L_{t,x}^{2+\frac{4}{n}}} \leq C\||D|^{\frac{1}{2}}v^*(1)\|_{L^2_x}\|v^*\|^{\frac{4}{n}}_{L_{t,x}^{2+\frac{4}{n}}},$$

where we have used (5.5) for $s = \frac{1}{2}$ at the last step.

By combining (5.9), (5.10) and (5.16) we get:

$$\||D|^{\frac{1}{2}}v^*(1)\|_{L^2_x} = \||D|^{\frac{1}{2}}(z^*(1) + w^*(1))\|_{L^2_x}$$

$$\leq C\left(\||D|^{\frac{1}{2}}v^*(1)\|_{L^2_x}\|v^*\|^{\frac{4}{n}}_{L_{t,x}^{2+\frac{4}{n}}} + \|g\|_{L^2_{|x|}}\right),$$

that due to the smallness of $\|v^*\|_{L_{t,x}^{2+\frac{4}{n}}}$ implies:

$$\|e^{-\mathbf{i}\frac{|x|^2}{4}}v(1)\|_{\dot{H}_x^{\frac{1}{2}}} = \||D|^{\frac{1}{2}}v^*(1)\|_{L^2_x} \leq C\|g\|_{L^2_{|x|}}.$$

□

Proposition 5.2. *Let $u \in \mathcal{C}_t(H^1_x) \cap L^{2+\frac{4}{n}}_{t,x}$ be the unique solution to*

$$\mathbf{i}\partial_t u - \Delta u \pm u|u|^{\frac{4}{n}} = 0,$$
$$u(0) = f$$

where $f \in H^1_x$ and $\|f\|_{L^2_x} < \epsilon$, with $\epsilon > 0$ small enough. Then the following estimate holds:

$$\|f\|_{\dot{H}^{\frac{1}{2}}_x} \leq C\|u(1)\|_{\dot{H}^{\frac{1}{2}}_x} \tag{5.17}$$

where $C > 0$ is a constant that does not depend on f.

Proof. Notice that $w = |D|^{\frac{1}{2}}u$ satisfies:

$$\mathbf{i}\partial_t w - \Delta w \pm |D|^{\frac{1}{2}}(u|u|^{\frac{4}{n}}) = 0$$
$$w(1) = |D|^{\frac{1}{2}}u(1),$$

then we can combine Strichartz inequalities with (5.1) in order to get:

$$\begin{aligned}\|w\|_{L^2_t L^{\frac{2n}{n-2}}_x} &\leq C\left(\||D|^{\frac{1}{2}}u(1)\|_{L^2_x} + \||D|^{\frac{1}{2}}(u|u|^{\frac{4}{n}})\|_{L_t^{\frac{2n+4}{n+6}} L_x^{\frac{2n(n+2)}{n^2-4+4n}}}\right)\\ &\leq C\left(\||D|^{\frac{1}{2}}u(1)\|_{L^2_x} + \||D|^{\frac{1}{2}}u\|_{L^2_t L^{\frac{2n}{n-2}}_x}\||u|^{\frac{4}{n}}\|_{L^{\frac{n+2}{2}}_{t,x}}\right)\\ &= C\left(\||D|^{\frac{1}{2}}u(1)\|_{L^2_x} + \||D|^{\frac{1}{2}}u\|_{L^2_t L^{\frac{2n}{n-2}}_x}\|u\|^{\frac{4}{n}}_{L^{2+\frac{4}{n}}_{t,x}}\right).\end{aligned}$$

Due to the smallness of $\|u\|_{L^{2+\frac{4}{n}}_{t,x}}$ (that comes from the smallness assumption done on f) we get

$$\||D|^{\frac{1}{2}}u\|_{L^2_t L^{\frac{2n}{n-2}}_x} = \|w\|_{L^2_t L^{\frac{2n}{n-2}}_x} \leq C\||D|^{\frac{1}{2}}u(1)\|_{L^2_x}. \tag{5.18}$$

By using again Strichartz estimates with a different choice of the parameters p, q we get

$$\begin{aligned}\|w\|_{L^\infty_t L^2_x} &\leq C\left(\||D|^{\frac{1}{2}}u(1)\|_{L^2_x} + \||D|^{\frac{1}{2}}(u|u|^{\frac{4}{n}})\|_{L_t^{\frac{2n+4}{n+6}} L_x^{\frac{2n(n+2)}{n^2-4+4n}}}\right)\\ &\leq C\left(\||D|^{\frac{1}{2}}u(1)\|_{L^2_x} + \||D|^{\frac{1}{2}}u\|_{L^2_t L^{\frac{2n}{n-2}}_x}\||u|^{\frac{4}{n}}\|_{L^{\frac{n+2}{2}}_{t,x}}\right)\end{aligned}$$

that is equivalent to

$$\||D|^{\frac{1}{2}}u\|_{L^\infty_t L^2_x} \leq C\left(\||D|^{\frac{1}{2}}u(1)\|_{L^2_x} + \||D|^{\frac{1}{2}}u\|_{L^2_t L^{\frac{2n}{n-2}}_x}\|u\|^{\frac{4}{n}}_{L^{2+\frac{4}{n}}_{t,x}}\right).$$

By combining this estimate with (5.18) we get

$$\||D|^{\frac{1}{2}}u\|_{L^\infty_t L^2_x} \leq C\||D|^{\frac{1}{2}}u(1)\|_{L^2_x}$$

and in particular

$$\begin{aligned}\|f\|_{\dot{H}^{\frac{1}{2}}_x} = \|u(0)\|_{\dot{H}^{\frac{1}{2}}_x} &\leq \||D|^{\frac{1}{2}}u\|_{L^\infty_t L^2_x}\\ &\leq C\||D|^{\frac{1}{2}}u(1)\|_{L^2_x} = \|u(1)\|_{\dot{H}^{\frac{1}{2}}_x}.\end{aligned}$$

The proof is complete.

$\square$

Proof of theorem 1.6. Let us recall that (2.7) implies the following identity:

$$\left\|\frac{x}{t}u(t) - 2\mathbf{i}\nabla u(t)\right\|_{L^2_x}^2 \pm \frac{2n}{n+2}\int |u(t)|^{2+\frac{4}{n}}dx = \frac{1}{t^2}\|f\|^2_{L^2_{|x|^2}}. \tag{5.19}$$

Since $u \in \mathcal{C}_t(H^1_x)$ it is meaningful to consider the trace $u(\bar{t}) \in H^1_x$ for any $\bar{t} \in \mathbf{R}$ and since by assumption $u \in L^{2+\frac{4}{n}}_{t,x}$, we can deduce that there exists a sequence $t_k \to \infty$ such that:

$$\int |u(t_k)|^{2+\frac{4}{n}}dx \to 0 \text{ as } k \to \infty. \tag{5.20}$$

By combining this fact with (5.19) we get:

$$\lim_{k\to\infty}\left\|\frac{x}{t_k}u(t_k) - 2\mathbf{i}\nabla u(t_k)\right\|_{L^2_x}^2 = 0. \tag{5.21}$$

We split now the proof in two parts.

Construction of g and proof of (1.12)

Let us recall that the conformal transfomation of $u(t,x)$:

$$\tilde{u}(t,x) = \frac{1}{t^{\frac{n}{2}}}e^{\frac{\mathbf{i}|x|^2}{4t}}u\left(\frac{1}{t},\frac{x}{t}\right),$$

satisfies the Cauchy problem (1.5) under the initial condition

$$\tilde{u}(1) = e^{\frac{\mathbf{i}|x|^2}{4}}u\left(1\right)$$

and in particular

$$\|\tilde{u}(1)\|_{L^2_x} = \|u(1)\|_{L^2_x} = \|f\|_{L^2_x} < \epsilon, \tag{5.22}$$

where we have used the conservation of the charge for the unique solution to (1.2) (see (2.4)).

Then $\tilde{u}$ satisfies the following Cauchy problem

$$\mathbf{i}\partial_t\tilde{u} + \Delta\tilde{u} \mp \tilde{u}|\tilde{u}|^{\frac{4}{n}} = 0,$$
$$\tilde{u}(1) \in L^2_x \text{ and } \|\tilde{u}(1)\|_{L^2_x} < \epsilon, (t,x) \in (0,\infty)\times\mathbf{R}^n.$$

Due to the global well - posedness of this Cauchy problem (see theorem 5.1) we deduce that $\tilde{u}$ can be extended as a solution to the whole space $\mathbf{R}\times\mathbf{R}^n$.

Moreover this extension belongs to the functional space $\mathcal{C}_t(L^2_x)\cap L^{2+\frac{4}{n}}_{t,x}$ and in particular it is well defined one unique $g \in L^2_x$ such that the following limit exists:

$$\lim_{t\to 0}\tilde{u}(t,x) = g \in L^2_x. \tag{5.23}$$

Hence $\tilde{u}(t,x)$ satisfies the following Cauchy problem

$$\mathbf{i}\partial_t\tilde{u} + \Delta\tilde{u} \mp \tilde{u}|\tilde{u}|^{\frac{4}{n}} = 0, \tag{5.24}$$
$$\tilde{u}(0) = g.$$

Due to (5.23) we can deduce that

$$\lim_{t\to 0}\left\|u\left(\frac{1}{t},x\right) - t^{\frac{n}{2}}e^{-\mathbf{i}t\frac{|x|^2}{4}}g\left(tx\right)\right\|_{L^2_x} = \lim_{t\to 0}\left\|\frac{1}{t^{\frac{n}{2}}}e^{\mathbf{i}\frac{|x|^2}{4t}}u\left(\frac{1}{t},\frac{x}{t}\right) - g\left(x\right)\right\|_{L^2_x} = 0,$$

and in particular

$$\lim_{t\to\infty}\left\|u(t)-\frac{1}{t^{\frac{n}{2}}}e^{-\mathbf{i}\frac{|x|^2}{4t}}g\left(\frac{x}{t}\right)\right\|_{L^2_x}=0. \tag{5.25}$$

On the other hand, since $\tilde{u}(t,x)$ satisfies (5.24), we can apply (5.2) in order to get:

$$\|e^{-\mathbf{i}\frac{|x|^2}{4}}\tilde{u}(1)\|^2_{\dot{H}^{\frac{1}{2}}_x}\leq C\int|x||g(x)|^2dx,$$

that due to the definition of $\tilde{u}(t,x)$ is equivalent to

$$\|u(1)\|^2_{\dot{H}^{\frac{1}{2}}_x}\leq C\int|x||g(x)|^2dx.$$

We can then combine this estimate with (5.17) in order to get (1.12).

Proof of (1.11)

Due to (5.21) we can deduce that

$$\lim_{k\to\infty}\left(\int\bar{u}(t_k)\nabla u(t_k)\cdot\nabla\psi\ dx+\frac{\mathbf{i}}{2t_k}\int|x||u(t_k)|^2\partial_{|x|}\psi\ dx\right)=0 \tag{5.26}$$

and then

$$\limsup_{t\to\infty}\left(-\mathcal{I}m\int\bar{u}(t)\nabla u(t)\cdot\nabla\psi\ dx\right) \tag{5.27}$$
$$\geq\frac{1}{2}\liminf_{k\to\infty}\int|x||u(t_k)|^2\partial_{|x|}\psi\ \frac{dx}{t_k}.$$

Let us fix now a real number $R>0$ and let us notice that due to the non - negativity assumption done on $\partial_{|x|}\psi$ we get:

$$\int|x||u(t_k)|^2\partial_{|x|}\psi\ \frac{dx}{t_k}\geq\int_{|x|\leq Rt_k}|x||u(t_k)|^2\partial_{|x|}\psi\ \frac{dx}{t_k} \tag{5.28}$$
$$=\int_{|x|\leq Rt_k}|x|\left(|u(t_k)|^2-\frac{1}{t_k^n}\left|g\left(\frac{x}{t_k}\right)\right|^2\right)\partial_{|x|}\psi\ \frac{dx}{t_k}$$
$$+\int_{|x|\leq Rt_k}|x|\left[(\partial_{|x|}\psi-\psi'(\infty)\right]\left|g\left(\frac{x}{t_k}\right)\right|^2\frac{dx}{t_k^{n+1}}$$
$$+\psi'(\infty)\int_{|x|\leq Rt_k}|x|\left|g\left(\frac{x}{t_k}\right)\right|^2\frac{dx}{t_k^{n+1}}$$

where g is the function constructed in the previous step.

Notice that the following estimate is trivial:

$$\int_{|x|\leq Rt_k}|x|\left(|u(t_k)|^2-\frac{1}{t_k^n}\left|g\left(\frac{x}{t_k}\right)\right|^2\right)\partial_{|x|}\psi\ \frac{dx}{t_k} \tag{5.29}$$
$$\leq R\|\partial_{|x|}\psi\|_{L^\infty_x}\int_{|x|\leq Rt_k}\left||u(t_k)|^2-\frac{1}{t_k^n}\left|g\left(\frac{x}{t_k}\right)\right|^2\right|dx\to 0\text{ as }k\to\infty,$$

where in the last step we have used (5.25).

On the other hand due to the change of variable formula we can prove that:

$$\left|\int_{|x|\leq Rt_k} |x| \left[(\partial_{|x|}\psi - \psi'(\infty)\right] \left|g\left(\frac{x}{t_k}\right)\right|^2 \frac{dx}{t_k^{n+1}}\right|$$

$$\leq R\int_{|x|\leq R} \left|\partial_{|x|}\psi(t_k x) - \psi'(\infty)\right| |g(x)|^2 dx,$$

that in conjunction with the dominated convergence theorem and with assumption (1.10) implies:

$$\lim_{k\to\infty} \int_{|x|\leq Rt_k} |x| \left[(\partial_{|x|}\psi - \psi'(\infty)\right] \left|g\left(\frac{x}{t_k}\right)\right|^2 \frac{dx}{t_k^{n+1}} = 0. \tag{5.30}$$

Due again to the change of variable formula we get

$$\int_{|x|\leq Rt_k} |x| \left|g\left(\frac{x}{t_k}\right)\right|^2 \frac{dx}{t_k^{n+1}} = \int_{|x|\leq R} |x||g(x)|^2 dx,$$

and in particular

$$\lim_{k\to\infty} \psi'(\infty) \int_{|x|\leq Rt_k} |x| \left|g\left(\frac{x}{t_k}\right)\right|^2 \frac{dx}{t_k^{n+1}} = \psi'(\infty)\int_{|x|\leq R} |x||g(x)|^2 dx. \tag{5.31}$$

By combining (5.29),(5.30), (5.31) and (5.28) we can deduce:

$$\liminf_{k\to\infty} \int |x||u(t_k)|^2 \partial_{|x|}\psi \ \frac{dx}{t_k} \tag{5.32}$$

$$\geq \psi'(\infty) \int_{|x|\leq R} |x||g(x)|^2 dx \ \forall R > 0.$$

Since $R > 0$ is arbitrary, we can combine (5.27) with (5.32), in order to deduce (1.11).

□

Proof of theorem 1.5 The proof is similar to the one of theorem 1.3. Let g denotes the function constructed in theorem 1.6, then we shall show that:

$$\liminf_{t\to\infty} \int |x||u(t)|^2 \frac{dx}{t} \geq \int_{|x|\leq R} |x||g(x)|^2 dx \ \forall R > 0. \tag{5.33}$$

In fact we have:

$$\int |x||u(t)|^2 \frac{dx}{t} \geq \int_{|x|\leq Rt} |x||u(t)|^2 \frac{dx}{t} \tag{5.34}$$

$$= \int_{|x|\leq Rt} |x| \left[|u(t)|^2 - \frac{1}{t^n}\left|g\left(\frac{x}{t}\right)\right|^2\right] \frac{dx}{t} + \int_{|x|\leq Rt} |x| \left|g\left(\frac{x}{t}\right)\right|^2 \frac{dx}{t^{n+1}} \ \forall R > 0.$$

Notice that the following estimate is trivial:

$$\int_{|x|\leq Rt} |x| \left(|u(t)|^2 - \frac{1}{t^n}\left|g\left(\frac{x}{t}\right)\right|^2\right) \frac{dx}{t} \tag{5.35}$$

$$\leq R\int_{|x|\leq Rt} \left||u(t)|^2 - \frac{1}{t^n}\left|g\left(\frac{x}{t}\right)\right|^2\right| dx \to 0 \text{ as } t\to\infty,$$

where in the last step we have used (5.25).

Due again to the change of variable formula we get

$$\int_{|x|\leq Rt} |x| \left| g\left(\frac{x}{t}\right)\right|^2 \frac{dx}{t^{n+1}} = \int_{|x|\leq R} |x||g(x)|^2 dx. \tag{5.36}$$

Hence (5.33) follows by combining (5.34), (5.35), (5.36).

In particular if

$$\liminf_{t\to\infty} \int |x||u(t)|^2 \frac{dx}{t} = 0,$$

then (5.33) implies $g \equiv 0$, and in turn due to (5.25) it gives $\lim_{t\to\infty} \|u(t)\|_{L^2_x} = 0$. By combining this fact with the conservation of the charge (2.4) we get $f \equiv 0$ and hence $u \equiv 0$.

□

6. Proof of theorem 1.4

We shall follow basically the strategy that has been used in the proof of theorems 1.1 and 1.2. More precisely we multiply (1.2) by the quantity given in (4.1), we integrate on the strip $(0,T)\times \mathbf{R}^n$ and with elementary computations we get:

$$\int_0^T \int \left[\nabla \bar{u} D^2\psi \nabla u - \frac{1}{4}|u|^2 \Delta^2 \psi \pm \frac{1}{n+2}|u|^{2+\frac{4}{n}} \Delta\psi\right] dxdt \tag{6.1}$$
$$= -\frac{1}{2}\mathcal{I}m \int \bar{u}(T)\nabla u(T)\cdot \nabla\psi\ dx + \frac{1}{2}\mathcal{I}m \int \bar{f}\ \nabla f \cdot \nabla \psi\ dx.$$

The following lemma will be very important in order to prove the r.h.s. estimate in (1.8).

Lemma 6.1. *Assume that $u \in \mathcal{C}_t(H^1_x) \cap L^{2+\frac{4}{n}}_{t,x}$ solves (1.2) with $f \in H^1_x$ and $\|f\|_{L^2_x} < \epsilon$, where $\epsilon > 0$ is a small number. Assume moreover that ϕ satisfies the same assumption as in proposition 7.1, then there exists a constant $C > 0$ that does not depend on $R > 0$ and such that following estimates hold:*

$$\left|\int u(t)\nabla \bar{u}(t)\cdot \nabla \phi_R\ dx\right| \leq C\|f\|^2_{\dot{H}^{\frac{1}{2}}_x}\ \forall t \in \mathbf{R}, R > 0 \tag{6.2}$$

and

$$\int\int |u|^{2+\frac{4}{n}} |\Delta\phi_R|\ dxdt \leq CR^{\frac{2}{n}-1}\epsilon^2 \|f\|^{\frac{4}{n}}_{\dot{H}^{\frac{1}{2}}_x}\ \forall R > 0, \tag{6.3}$$

where $\phi_R = R\phi\left(\frac{x}{R}\right)$.

Proof.

Proof of (6.2)

The proof of (6.2) is similar to the proof of (4.4) provided that we are able to show the following a - priori bound:

$$\|u(t)\|_{\dot{H}^{\frac{1}{2}}_x} \leq C\|f\|_{\dot{H}^{\frac{1}{2}}_x}\ \forall t \in \mathbf{R}, \tag{6.4}$$

where u and f are as in the statement.

In order to prove this inequality we notice that:

$$\mathbf{i}\partial_t(|D|^{\frac{1}{2}}u) - \Delta(|D|^{\frac{1}{2}}u) = \mp |D|^{\frac{1}{2}}(u|u|^{\frac{4}{n}}),$$
$$|D|^{\frac{1}{2}}u(0) = |D|^{\frac{1}{2}} f$$

hence by combining Strichartz estimates with (5.1), and following the proof of (5.5), we can get the following estimate:

$$\| |D|^{\frac{1}{2}} u\|_{L^2_t L^{\frac{2n}{n-2}}_x} \leq C\|f\|_{\dot{H}^{\frac{1}{2}}_x}. \tag{6.5}$$

On the other hand by combining again Strichartz estimates with (5.1) we get:

$$\begin{aligned} \| |D|^{\frac{1}{2}} u\|_{L^\infty_t L^2_x} &\leq C \left(\| |D|^{\frac{1}{2}} f\|_{L^2_x} + \| |D|^{\frac{1}{2}} (u|u|^{\frac{4}{n}})\|_{L_t^{\frac{2n+4}{n+6}} L_x^{\frac{2n(n+2)}{n^2-4+4n}}} \right) \\ &\leq C \left(\| |D|^{\frac{1}{2}} f\|_{L^2_x} + \| |D|^{\frac{1}{2}} u\|_{L^2_t L^{\frac{2n}{n-2}}_x} \|u\|^{\frac{4}{n}}_{L^{2+\frac{4}{n}}_{t,x}} \right) \\ &\leq C \left(\| |D|^{\frac{1}{2}} f\|_{L^2_x} + \| |D|^{\frac{1}{2}} f\|_{L^2_x} \|u\|^{\frac{4}{n}}_{L^{2+\frac{4}{n}}_{t,x}} \right), \end{aligned} \tag{6.6}$$

where we have used (6.5) at the last step.

Hence we can deduce (6.4) by using the boundedness of $\|u\|_{L^{2+\frac{4}{n}}_{t,x}}$, that in turn comes from the smallness assumption done on f.

Proof of (6.3)

Since ϕ satisfies proposition 7.1, it is easy to deduce that

$$|\Delta\phi| \leq \frac{C}{1+|x|} \ \forall x \in \mathbf{R}^n. \tag{6.7}$$

Next we shall need the following a - priori estimates:

$$\|u\|_{L^2_t L^{\frac{2n}{n-2}}_x} \leq C\|f\|_{L^2_x} \tag{6.8}$$

and

$$\|u\|_{L^\infty_t \dot{H}^{\frac{1}{2}}_x} \leq C\|f\|_{\dot{H}^{\frac{1}{2}}_x}. \tag{6.9}$$

Notice that (6.9) is equivalent to (6.4), then we shall show (6.8).

Since u solves (1.2) we are in position to apply the Strichartz estimates in order to deduce:

$$\begin{aligned} \|u\|_{L^2_t L^{\frac{2n}{n-2}}_x} &\leq C \left(\|f\|_{L^2_x} + \|u|u|^{\frac{4}{n}}\|_{L_t^{\frac{2n+4}{n+6}} L_x^{\frac{2n(n+2)}{n^2-4+4n}}} \right) \\ &\leq C \left(\|f\|_{L^2_x} + \|u\|_{L^2_t L^{\frac{2n}{n-2}}_x} \| |u|^{\frac{4}{n}}\|_{L^{\frac{n+2}{2}}_{t,x}} \right) \\ &= C \left(\|f\|_{L^2_x} + \|u\|_{L^2_t L^{\frac{2n}{n-2}}_x} \|u\|^{\frac{4}{n}}_{L^{2+\frac{4}{n}}_{t,x}} \right), \end{aligned}$$

that due to the smallness of $\|u\|^{\frac{4}{n}}_{L^{2+\frac{4}{n}}_{t,x}}$ (that depends as usual on the smallness assumption done on f) implies (6.8).

The Hölder inequality implies the following chain of estimates:

$$\int\int |u|^{2+\frac{4}{n}} |\Delta\phi_R| \ dxdt = \frac{1}{R}\int\int |u|^{2+\frac{4}{n}} \left|\Delta\phi\left(\frac{x}{R}\right)\right| dxdt \tag{6.10}$$

$$\leq \frac{1}{R}\left(\int \left|\Delta\phi\left(\frac{x}{R}\right)\right|^{\frac{n^2}{2}} dx\right)^{\frac{2}{n^2}} \int \|u(t)\|^{2+\frac{4}{n}}_{L_x^{\frac{2n(n+2)}{n^2-2}}} dt$$

$$\leq \frac{1}{R}\left(\int \left|\Delta\phi\left(\frac{x}{R}\right)\right|^{\frac{n^2}{2}} dx\right)^{\frac{2}{n^2}} \int \|u(t)\|^{\theta(2+\frac{4}{n})}_{L_x^{\frac{2n}{n-1}}} \|u(t)\|^{(1-\theta)(2+\frac{4}{n})}_{L_x^{\frac{2n}{n-2}}} dt,$$

where

$$\frac{n^2-2}{2n(n+2)} = \frac{\theta(n-1)}{2n} + \frac{(1-\theta)(n-2)}{2n},$$

or equivalently $\theta = \frac{2}{n+2}$.

Then the previous estimate becomes

$$\int\int |u|^{2+\frac{4}{n}} |\Delta\phi_R| \, dxdt$$

$$\leq \frac{1}{R}\left(\int \left|\Delta\phi\left(\frac{x}{R}\right)\right|^{\frac{n^2}{2}} dx\right)^{\frac{2}{n^2}} \int \|u(t)\|^{\frac{4}{n}}_{L_x^{\frac{2n}{n-1}}} \|u(t)\|^2_{L_x^{\frac{2n}{n-2}}} dt$$

$$\leq \frac{1}{R}\left(\int \left|\Delta\phi\left(\frac{x}{R}\right)\right|^{\frac{n^2}{2}} dx\right)^{\frac{2}{n^2}} \|u(t)\|^{\frac{4}{n}}_{L_t^\infty L_x^{\frac{2n}{n-1}}} \|u(t)\|^2_{L_t^2 L_x^{\frac{2n}{n-2}}},$$

that due to the Sobolev embedding

$$\dot H_x^{\frac{1}{2}} \to L_x^{\frac{2n}{n-1}} \tag{6.11}$$

implies:

$$\int\int |u|^{2+\frac{4}{n}} |\Delta\phi_R| \, dxdt \tag{6.12}$$

$$\leq \frac{1}{R}\left(\int \left|\Delta\phi\left(\frac{x}{R}\right)\right|^{\frac{n^2}{2}} dx\right)^{\frac{2}{n^2}} \|u(t)\|^{\frac{4}{n}}_{L_t^\infty \dot H_x^{\frac{1}{2}}} \|u(t)\|^2_{L_t^2 L_x^{\frac{2n}{n-2}}}.$$

Notice that due to (6.7) we get

$$\left\|\Delta\phi\left(\frac{x}{R}\right)\right\|_{L_x^{\frac{n^2}{2}}} \leq CR^{\frac{2}{n}}\left(\int \frac{1}{(1+|x|)^{\frac{n^2}{2}}} \, dx\right)^{\frac{2}{n^2}} \quad \forall R>0.$$

By combining now this estimate with (6.8), (6.9), and (6.12) we deduce that

$$\int\int |u|^{2+\frac{4}{n}} |\Delta\phi_R| \, dxdt \tag{6.13}$$

$$\leq CR^{\frac{2}{n}-1}\|f\|^{\frac{4}{n}}_{\dot H_x^{\frac{1}{2}}} \|f\|^2_{L_x^2} \leq CR^{\frac{2}{n}-1}\epsilon^2 \|f\|^{\frac{4}{n}}_{\dot H_x^{\frac{1}{2}}} \; \forall R>0.$$

□

We shall need also the following

Lemma 6.2. *Let $u \in \mathcal{C}_t(L^2_x) \cap L^{2+\frac{4}{n}}_{t,x}$ be the unique solution to* (1.2) *where $\|f\|_{L^2_x} < \epsilon$, with $\epsilon > 0$ small, then:*

$$\lim_{R\to\infty} \int\int |u|^2 |\Delta^2\phi_R| \, dxdt = 0 \tag{6.14}$$

and

$$\lim_{R\to\infty}\int\int |u|^{2+\frac{4}{n}}|\Delta\phi_R|\; dxdt = 0, \tag{6.15}$$

where $\phi \in C^\infty(\mathbf{R}^n)$ *is a radially symmetric function such that*

$$|\partial_{|x|}\phi| \le C, |\Delta^2\phi| \le \frac{C}{(1+|x|)^3}\ \forall x \in \mathbf{R}^n$$

and $\phi_R = R\phi\left(\frac{x}{R}\right)$.

Proof.

Proof of (6.14)

Let us notice that since u solves (1.2) we can apply Strichartz estimates in order to get:

$$\|u\|_{L^2_t L_x^{\frac{2n}{n-2}}} \le C\left(\|f\|_{L^2_x} + \||u|^{1+\frac{4}{n}}\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}}\right) \tag{6.16}$$

$$= C\left(\|f\|_{L^2_x} + \|u\|^{1+\frac{4}{n}}_{L^{2+\frac{4}{n}}_{t,x}}\right) < \infty.$$

On the other hand the Hölder inequality implies:

$$\int\int |u|^2|\Delta^2\phi_R(x)|\; dxdt \le \left[\int\left(\int |u|^{\frac{2n}{n-2}}\; dx\right)^{\frac{n-2}{n}} dt\right]\left(\int |\Delta^2\phi_R(x)|^{\frac{n}{2}}\; dx\right)^{\frac{2}{n}}$$

$$\le C\|u\|^2_{L^2_t L_x^{\frac{2n}{n-2}}}\left(\int \frac{1}{(R+|x|)^{\frac{3n}{2}}}\; dx\right)^{\frac{2}{n}} \to 0 \text{ as } R\to\infty,$$

where we have used in the last step the fact that $\|u\|_{L^2_t L_x^{\frac{2n}{n-2}}} < \infty$ that comes from (6.16).

Proof of (6.15)

By using the Hölder inequality we get:

$$\int\int |u|^{2+\frac{4}{n}}|\Delta\phi_R|\; dxdt \le \|\Delta\phi_R\|_{L^\infty_x}\int\int |u|^{2+\frac{4}{n}}\; dxdt$$

$$= \frac{1}{R}\|\Delta\phi\|_{L^\infty_x}\|u\|^{2+\frac{4}{n}}_{L^{2+\frac{4}{n}}_{t,x}} \to 0 \text{ as } R\to\infty.$$

□

Proof of theorem 1.4

Let us first prove the estimate

$$\sup_{R>R_0}\frac{1}{R}\int_0^\infty\int_{|x|<R}|\nabla u|^2\; dxdt \tag{6.17}$$

$$\leq C\left(\|f\|^2_{\dot{H}^{\frac{1}{2}}_x}+\frac{1}{R_0^{1-\frac{2}{n}}}\|f\|^{\frac{4}{n}}_{\dot{H}^{\frac{1}{2}}_x}\right)\ \forall R_0>0.$$

Notice that (6.1) implies:

$$\int_0^T\int\left[\nabla\bar{u}D^2\psi\nabla u-\frac{1}{4}|u|^2\Delta^2\psi\right]dxdt=$$

$$\mp\frac{1}{n+2}\int_0^T\int|u|^{2+\frac{4}{n}}\Delta\psi\ dxdt$$

$$-\frac{1}{2}\mathcal{I}m\int\bar{u}(T)\nabla u(T)\cdot\nabla\psi\ dx+\frac{1}{2}\mathcal{I}m\int\bar{f}\ \nabla f\cdot\nabla\psi\ dx.$$

If we choose in the previous identity $\psi=R\phi\left(\frac{x}{R}\right)$ (with $R>R_0$) where ϕ is as in proposition 7.1, and if we recall (6.2),(6.3), then we can deduce (6.17). Notice that (6.17) trivially implies

$$\limsup_{R\to\infty}\frac{1}{R}\int\int_{B_R}|\nabla u|^2\ dxdt\leq C\|f\|^2_{\dot{H}^{\frac{1}{2}}_x}.$$

Next we shall prove

$$\liminf_{R\to\infty}\frac{1}{R}\int\int_{B_R}|\nabla u|^2\ dxdt\geq c\|f\|^2_{\dot{H}^{\frac{1}{2}}_x}\tag{6.18}$$

and it will be sufficient to complete the proof of the theorem.

In fact the proof of (6.18) can be done by exploiting the identity (6.1), where we choose the function ψ to be equal to the functions ϕ_R used in the proof of theorem 1.2.

Then the argument follows as in the proof of theorem 1.2 with some minor changes. In fact it is sufficient to replace proposition 3.1 with theorem 1.6, and to use (6.14) and (6.15) instead of (4.11) and (4.12).

□

7. Appendix

In order to make this paper self - contained we shall give in this appendix a result already presented in [2]. More precisely we shall prove the existence of a test function ϕ with suitable properties that has been extensively used along this paper.

Proposition 7.1. *Assume that $n\geq 3$, then there exists a radially symmetric function $\phi:\mathbf{R}^n\to\mathbf{R}$ such that:*

1. $\phi(0)=\partial_{|x|}\phi(0)=0$ *and* $\partial^2_{|x|}\phi(0)>0$*;*
2. *for any $x\in\mathbf{R}^n$ we have*
$$\Delta^2\phi\leq 0\ \forall x\in\mathbf{R}^n;$$
3. $\partial_{|x|}\phi,\partial^2_{|x|}\phi>0\ \forall x\in\mathbf{R}^n\setminus\{0\}$;
4. *there exists $C>0$ such that*
$$\partial_{|x|}\phi,|x|\partial^2_{|x|}\phi\leq C\ \forall x\in\mathbf{R}^n;$$
5. *the following limit exists*
$$\lim_{|x|\to\infty}\partial_{|x|}\phi\in(0,\infty).$$

In particular we have:

$$|\Delta\phi| \leq \frac{C}{1+|x|}\ \forall x \in \mathbf{R}^n, \tag{7.1}$$

$$\nabla u D^2\phi\nabla\bar{u} \geq C(|\partial^2_{|x|}u|^2 + |\nabla_\tau u|^2)\ \forall x \in \mathbf{R}^n\ s.t.\ |x| < 1, \tag{7.2}$$

and

$$\nabla u D^2\phi\nabla\bar{u} \geq C\frac{|\nabla_\tau u|^2}{|x|}\ \forall x \in \mathbf{R}^n\ s.t.\ |x| > 1, \tag{7.3}$$

for any function u.

Proof.

It is easy to show that (7.1) follows by writing the laplacian in polar coordinates, while (7.2) and (7.3) follow from the identity (4.17).

Next we shall focus on the construction of ϕ that satisfies $(1), (2), (3), (4)$.

The main strategy is to solve the following equation:

$$-\Delta^2\phi = h_\eta(|x|)$$

where

$$h_\eta(|x|) = \chi_{\{|x|<1\}} + \frac{\eta}{|x|^3}\chi_{\{|x|>1\}} \tag{7.4}$$

and $\eta \geq 0$ is a suitable parameter that will be choosen later.

The equation $-\Delta^2\phi = h_\eta$ can be written in polar coordinates in the following equivalent way:

$$-r^{-(n-1)}\partial_r(r^{n-1}\partial_r(r^{-(n-1)}\partial_r(r^{n-1}\partial_r\phi(r)))) = h_\eta(r).$$

By integrating directly this equation we get:

$$\partial_r\phi = -\frac{1}{r^{n-1}}\int_0^r u^{n-1}\int_0^u \frac{1}{s^{n-1}}\int_0^s t^{n-1}h(t)dt + \lambda r, \tag{7.5}$$

where $\lambda \in \mathbf{R}$ is a generic number that will be choosen later.

Next we split the proof in two cases.

First case: $n = 3$

Let us choose h_η as in (7.4) with $\eta = 0$, in this way by an explicit integration we get:

$$\partial_r\phi = \lambda r - \frac{r^3}{30}\ \forall\ 0 < r < 1$$

$$\partial_r\phi = \lambda r + \frac{1}{6} - \frac{r}{6} - \frac{1}{30r^2}\ \forall\ r > 1.$$

In particular if we choose $\lambda = \frac{1}{6}$ then we deduce that $\partial_r\phi$ satisfies the desired assumptions. Moreover with the previous choice of λ we have:

$$\partial^2_r\phi = \frac{1}{6} - \frac{r^2}{10}\ \forall\ 0 < r < 1$$

$$\partial^2_r\phi = \frac{1}{15r^3}\ \forall\ r > 1.$$

It is now easy to check that all the properties required to ϕ are fulfilled.

Second case: $n > 4$

In this case we choose $\eta > 0$ in (7.4). The precise value of η will be choosen later.

An explicit integration of (7.5) gives:

$$\partial_r \phi = \lambda r - \frac{r^3}{2n(n+2)} \ \forall\ 0 < r < 1$$

$$\partial_r \phi = \lambda r + \frac{\eta}{(n-1)(n-3)} - \frac{(1+2\eta)n - (3+6\eta)}{2n(n-2)(n-3)} r - \frac{\eta n - n + 3}{2n(n-2)(n-3)} \frac{1}{r^{n-3}}$$
$$- \left[\frac{(1-\eta)n^4 + (3\eta - 6)n^3 + (11+4\eta)n^2 - (12\eta+6)n}{2n^2(n-1)(n-2)(n-3)(n+2)} \right] \frac{1}{r^{n-1}} \ \forall\ r > 1.$$

Notice that if we choose

$$\lambda = \frac{(1+2\eta)n - (3+6\eta)}{2n(n-2)(n-3)} \tag{7.6}$$

then we get

$$\partial_r^2 \phi = \lambda - \frac{3r^2}{2n(n+2)} \ \forall\ 0 < r < 1$$

$$\partial_r^2 \phi = \frac{\eta n - n + 3}{2n(n-2)} \frac{1}{r^{n-2}}$$
$$+ \left[\frac{(1-\eta)n^4 + (3\eta - 6)n^3 + (11+4\eta)n^2 - (12\eta+6)n}{2n^2(n-2)(n-3)(n+2)} \right] \frac{1}{r^{n}} \ \forall\ r > 1.$$

It is easy to verify that all the properties required to ϕ will be fulfilled provided that we can choose λ and η that satisfy (7.6) and moreover

$$\lambda > \frac{3}{2n(n+2)},\ \eta n - n + 3 > 0,$$

$$(1-\eta)n^4 + (3\eta - 6)n^3 + (11+4\eta)n^2 - (12\eta+6)n > 0,$$

$$\frac{\eta}{(n-1)(n-3)} > \frac{\eta n - n + 3}{2n(n-2)(n-3)}$$
$$+ \frac{(1-\eta)n^4 + (3\eta - 6)n^3 + (11+4\eta)n^2 - (12\eta+6)n}{2n^2(n-1)(n-2)(n-3)(n+2)}.$$

Explicit computations show that the previous conditions are equivalent to look for a suitbale $\eta > 0$ such that:

$$Max \left\{ \frac{n-3}{n}, \frac{-2n^2 + 8n - 6}{n^3 - 2n^2 - 5n + 6} \right\} < \eta < \frac{n^3 - 6n^2 + 11n - 6}{(n-2)(n+2)(n-3)}.$$

An elementary computation shows that

$$\frac{-2n^2 + 8n - 6}{n^3 - 2n^2 - 5n + 6} < 0$$

for $n > 3$, it is then enough to verify that

$$\frac{n-3}{n} < \frac{n^3 - 6n^2 + 11n - 6}{(n-2)(n+2)(n-3)}.$$

It is easy to verify that this condition is fulfilled for any $n \geq 4$.

□

References

[1] *S. Agmon and L. Hörmander* Asymptotic properties of solutions of differential equations with simple characteristics, J. Analyse Math., vol.30, 1976, pp. 1-38.

[2] *J.A. Barcelo, A. Ruiz and L. Vega* Some dispersive estimates for Schrödinger equations with repulsive potentials. Preprint.

[3] *N. Burq, F. Planchon, J. Stalker and S. Tahvildar-Zadeh* Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay, Indiana Univ. Math. J., vol. 53, 2004, (6), pp. 1665-1680.

[4] *T. Cazenave* Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[5] *T. Cazenave and F. Weissler* The Cauchy problem for the critical nonlinear Schrödinger equation in H^s, Nonlinear Anal. Theory, Methods & Applications. An International Multidisciplinary Journal. Series A: Theory and Methods, vol. 14, 1990, (10), pp. 807–836.

[6] *T. Cazenave and F. Weissler* Some remarks on the nonlinear Schrödinger equation in the subcritical case, New methods and results in nonlinear field equations (Bielefeld, 1987), Lecture Notes in Phys., vol. 347, pp. 59–69, Springer, Berlin, 1989.

[7] *P. Constantin and J.C. Saut* Local smoothing properties of Schrödinger equations, Indiana Univ. Math. J., vol. 38, 1989, (3), pp. 791-810.

[8] *M. Keel and T. Tao* Endpoint Strichartz estimates, Amer. J. Math., vol. 120, 1998, (5), pp. 955–980.

[9] *P. Sjölin* Regularity of solutions to the Schrödinger equation, Duke Math. J., vol. 55, 1987, (3), pp. 699-715.

[10] *Y. Tsutsumi* L^2-solutions for nonlinear Schrödinger equations and nonlinear groups, Funkcial. Ekvac., vol. 30, 1987, (1), pp. 115–125.

[11] *L. Vega* Schrödinger equations: pointwise convergence to the initial data, Proc. Amer. Math. Soc., vol. 102, 1988, (4), pp. 874-878.

[12] *L.Vega and N. Visciglia* On the local smoothing for the Schrödinger equation, to appear on Proc. Amer. Math. Soc.

Luis Vega, Universidad del Pais Vasco, Apdo. 64, 48080 Bilbao, Spain
E-mail address: mtpvegol@lg.ehu.es

Nicola Visciglia, Dipartimento di Matematica Università di Pisa, Largo B. Pontecorvo 5, 56100 Pisa, Italy
E-mail address: viscigli@dm.unipi.it